

# On Symmetries of Extremal Black Holes with One and Two Centers

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## ABSTRACT

After a brief introduction to the *Attractor Mechanism*, we review the appearance of groups of type  $E_7$  as generalized electric-magnetic duality symmetries in locally supersymmetric theories of gravity, with particular emphasis on the symplectic structure of fluxes in the background of extremal black hole solutions, with one or two centers. In the latter case, the role of an “horizontal” symmetry  $SL_h(2, \mathbb{R})$  is elucidated by presenting a set of two-centered relations governing the structure of two-centered invariant polynomials.

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# 1 Introduction

The *Attractor Mechanism* (AM) [1] governs the dynamics in the scalar manifold of Maxwell-Einstein (super)gravity theories. It keeps standing as a crucial fascinating key topic. Along the last years, a number of papers have been devoted to the investigation of attractor configurations of extremal black  $p$ -branes in diverse space-time dimensions; for some lists of Refs., see *e.g.* [2].

The AM is related to dynamical systems with fixed points, describing the equilibrium state and the stability features of the system under consideration<sup>1</sup>. When the AM holds, the particular property of the long-range behavior of the dynamical flows in the considered (dissipative) system is the following: in approaching the fixed points, properly named *attractors*, the orbits of the dynamical evolution lose all memory of their initial conditions, but however the overall dynamics remains completely deterministic.

The first example of AM in supersymmetric systems was discovered in the theory of static, spherically symmetric, asymptotically flat extremal dyonic black holes in  $N=2$  Maxwell-Einstein supergravity in  $d=4$  and 5 space-time dimensions (see the first two Refs. of [1]). In the following, we will briefly present some basic facts about the  $d=4$  case.

The multiplet content of a completely general  $N=2$ ,  $d=4$  supergravity theory is the following (see *e.g.* [3], and Refs. therein):

1. the *gravitational* multiplet

$$(V_\mu^a, \psi^A, \psi_A, A^0), \quad (1.1)$$

described by the *Vielbein* one-form  $V^a$  ( $a=0,1,2,3$ ) (together with the spin-connection one-form  $\omega^{ab}$ ), the  $SU(2)$  doublet of gravitino one-forms  $\psi^A, \psi_A$  ( $A=1,2$ , with the upper and lower indices respectively denoting right and left chirality, *i.e.*  $\gamma_5 \psi_A = -\gamma_5 \psi^A$ ), and the graviphoton one-form  $A^0$ ;

2.  $n_V$  *vector* supermultiplets

$$(A^I, \lambda^{iA}, \bar{\lambda}_{\bar{A}}^{\bar{i}}, z^i), \quad (1.2)$$

each containing a gauge boson one-form  $A^I$  ( $I=1,\dots,n_V$ ), a doublet of gauginos (zero-form spinors)  $\lambda^{iA}, \bar{\lambda}_{\bar{A}}^{\bar{i}}$ , and a complex scalar field (zero-form)  $z^i$  ( $i=1,\dots,n_V$ ). The scalar fields  $z^i$  can be regarded as coordinates on a complex manifold  $\mathcal{M}_{n_V}$  ( $\dim_{\mathbb{C}} \mathcal{M}_{n_V} = n_V$ ), which is actually a *special Kähler* manifold;

3.  $n_H$  *hypermultiplets*

$$(\zeta_\alpha, \zeta^\alpha, q^u), \quad (1.3)$$

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<sup>1</sup>We recall that a point  $x_{fix}$  where the phase velocity  $v(x_{fix})$  vanishes is called a *fixed* point, and it gives a representation of the considered dynamical system in its equilibrium state,

$$v(x_{fix}) = 0.$$

The fixed point is said to be an *attractor* of some motion  $x(t)$  if

$$\lim_{t \rightarrow \infty} x(t) = x_{fix}.$$

each formed by a doublet of zero-form spinors, that is the hyperinos  $\zeta_\alpha, \zeta^\alpha$  ( $\alpha = 1, \dots, 2n_H$ ), and four real scalar fields  $q^u$  ( $u = 1, \dots, 4n_H$ ), which can be considered as coordinates of a quaternionic manifold  $\mathcal{H}_{n_H}$  ( $\dim_{\mathbb{H}} \mathcal{H}_{n_H} = n_H$ ).

At least in absence of gauging and without quantum corrections, the  $n_H$  hypermultiplets are spectators in the AM. This can be understood by looking at the transformation properties of the Fermi fields: the hyperinos  $\zeta_\alpha, \zeta^\alpha$ 's transform independently on the vector fields, whereas the gauginos' supersymmetry transformations depend on the Maxwell vector fields. Consequently, the contribution of the hypermultiplets can be dynamically decoupled from the rest of the physical system; in particular, it is also completely independent from the evolution dynamics of the complex scalars  $z^i$ 's coming from the vector multiplets (*i.e.* from the evolution flow in  $\mathcal{M}_{n_V}$ ). Indeed, disregarding for simplicity's sake the fermionic and gauging terms, the supersymmetry transformations of gauginos and hyperinos respectively read (see *e.g.* [3], and Refs. therein)

$$\delta\lambda^{iA} = i\partial_\mu z^i \gamma^\mu \varepsilon^A + G_{\mu\nu}^{-i} \gamma^{\mu\nu} \varepsilon_B \epsilon^{AB}; \quad (1.4)$$

$$\delta\zeta_\alpha = i\mathcal{U}_u^{B\beta} \partial_\mu q^u \gamma^\mu \varepsilon^A \epsilon_{AB} \mathbb{C}_{\alpha\beta}. \quad (1.5)$$

(1.5) implies that the asymptotical configurations of the quaternionic hypermultiplets' scalars are *unconstrained*, and therefore they can vary continuously in the manifold  $\mathcal{H}_{n_H}$  of the related quaternionic non-linear sigma model.

Thus, as far as ungauged theories are concerned, for the treatment of AM one can restrict to consider  $N = 2$ ,  $d = 4$  Maxwell-Einstein supergravity, in which  $n_V$  vector multiplets (1.2) are coupled to the gravity multiplet (1.1). The relevant dynamical system to be considered is the one related to the radial evolution of the configurations of complex scalar fields of such  $n_V$  vector multiplets. When approaching the event horizon of the black hole, the scalars dynamically run into fixed points, taking values which are only function (of the ratios) of the electric and magnetic charges associated to Abelian Maxwell vector potentials under consideration.

The inverse distance to the event horizon is the fundamental evolution parameter in the dynamics towards the fixed points represented by the *attractor* configurations of the scalar fields. Such near-horizon configurations, which “attracts” the dynamical evolutive flows in  $\mathcal{M}_{n_V}$ , are completely independent on the initial data of such an evolution, *i.e.* on the spatial asymptotical configurations of the scalars. Consequently, for what concerns the scalar dynamics, the system completely loses memory of its initial data, because the dynamical evolution is “attracted” by some fixed configuration points, purely depending on the electric and magnetic charges.

In the framework of supergravity theories, extremal black holes can be interpreted as BPS (Bogomol'ny-Prasad-Sommerfeld)-saturated [4] interpolating metric singularities in the low-energy effective limit of higher-dimensional superstrings or  $M$ -theory [5]. Their asymptotically relevant parameters include the ADM mass [6], the electrical and magnetic charges (defined by integrating the fluxes of related field strengths over the 2-sphere at infinity), and the asymptotical values of the (dynamically relevant set of) scalar fields. The AM implies that the class of black holes under consideration loses all its “scalar hair” within the near-horizon geometry. This means that the extremal black hole solutions, in

the near-horizon limit in which they approach the Bertotti-Robinson  $AdS_2 \times S^2$  conformally flat metric [7], are characterized only by electric and magnetic charges, but not by the continuously-varying asymptotical values of the scalar fields.

An important progress in the geometric interpretation of the AM was achieved in the last Ref. of [1], in which the attractor near-horizon scalar configurations were related to the critical points of a suitably defined black hole effective potential function  $V_{BH}$ . In general,  $V_{BH}$  is a positive definite function of scalar fields and electric and magnetic charges, and its non-degenerate critical points in  $\mathcal{M}_{n_V}$

$$\forall i = 1, \dots, n_V, \frac{\partial V_{BH}}{\partial z^i} = 0 : V_{BH}|_{\frac{\partial V_{BH}}{\partial z}=0} > 0, \quad (1.6)$$

fix the scalar fields to depend only on electric and magnetic fluxes (charges). In the Einstein two-derivative approximation, the (semi)classical Bekenstein-Hawking entropy ( $S_{BH}$ ) - area ( $A_H$ ) formula [8] yields the (purely charge-dependent) black hole entropy  $S_{BH}$  to be

$$S_{BH} = \pi \frac{A_H}{4} = \pi V_{BH}|_{\frac{\partial V_{BH}}{\partial z}=0} = \pi \sqrt{|\mathcal{I}_4|}, \quad (1.7)$$

where  $\mathcal{I}_4$  is the unique independent invariant homogeneous polynomial (quartic in charges) in the relevant representation  $\mathbf{R}$  of  $G$  in which the charges sit (see Eq. (1.9) and discussion below). The last step of (1.7) does not apply to  $d = 4$  supergravity theories with quadratic charge polynomial invariant, namely to the  $N = 2$  *minimally coupled* sequence [9] and to the  $N = 3$  [12] theory; in these cases, in (1.7)  $\sqrt{|\mathcal{I}_4|}$  gets replaced by  $|\mathcal{I}_2|$ .

In presence of  $n = n_V + 1$  Abelian vector fields, the charge vector ( $\Lambda = 0, 1, \dots, n_V$ )

$$Q \equiv (p^\Lambda, q_\Lambda) \quad (1.8)$$

of magnetic ( $p^\Lambda$ ) and electric ( $q_\Lambda$ ) fluxes sits in a  $2n$ -dimensional representation  $\mathbf{R}$  of the  $U$ -duality<sup>2</sup> group  $G$ , defining the Gaillard-Zumino embedding [13] of  $G$  itself into  $Sp(2n, \mathbb{R})$ , which is the largest group acting linearly on the fluxes themselves:

$$G \xrightarrow{\mathbf{R}} Sp(2n, \mathbb{R}). \quad (1.9)$$

We consider here the (semi-)classical limit of large charges, also indicated by the fact that we consider  $Sp(2n, \mathbb{R})$ , and not  $Sp(2n, \mathbb{Z})$  (no Dirac-Schwinger-Zwanziger quantization condition is implemented on the fluxes themselves).

After [14, 15, 16], the  $\mathbf{R}$ -representation space of the  $U$ -duality group is known to exhibit a *stratification* into disjoint classes of orbits, which can be defined through invariant sets of constraints on the (lowest order, actually unique)  $G$ -invariant  $\mathcal{I}$  built out of the symplectic representation  $\mathbf{R}$ ; this will be reported in Sec. 3 It is here worth remarking the crucial distinction between the “large” orbits and “small” orbits. While the former have  $\mathcal{I} \neq 0$  and support an attractor behavior of the scalar flow in the near-horizon geometry of the extremal black hole background [1], for the latter the Attractor Mechanism does not hold, they have  $\mathcal{I} = 0$  and thus they correspond to solutions with vanishing Bekenstein-Hawking [8] entropy (*at least* at the Einsteinian two-derivative level).

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<sup>2</sup>Here  $U$ -duality is referred to as the “continuous” symmetries of [10]. Their discrete versions are the  $U$ -duality non-perturbative string theory symmetries introduced by Hull and Townsend [11].

## 2 $U$ -Duality and Groups of Type $E_7$

From the treatment above, the black hole entropy  $S_{BH}$  is invariant under the electric-magnetic duality, in which the non-compact  $U$ -duality group has a symplectic action both on the charge vector  $Q$  (1.8) and on the scalar fields (through the definition of a *flat* symplectic bundle [17] over the scalar manifold itself); see *e.g.* [18] for a review. The latter property makes relevant the mathematical notion of groups of type  $E_7$ .

The first axiomatic characterization of groups of type  $E_7$  through a module (irrep.) was given in 1967 by Brown [19]. A group  $G$  of type  $E_7$  is a Lie group endowed with a representation  $\mathbf{R}$  such that:

1.  $\mathbf{R}$  is *symplectic*, *i.e.* :

$$\exists! \mathbb{C}_{[MN]} \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R}; \quad (2.1)$$

(the subscripts “ $s$ ” and “ $a$ ” stand for symmetric and skew-symmetric throughout) in turn,  $\mathbb{C}_{[MN]}$  defines a non-degenerate skew-symmetric bilinear form (*symplectic product*); given two different charge vectors  $Q_1$  and  $Q_2$  in  $\mathbf{R}$ , such a bilinear form is defined as

$$\langle Q_1, Q_2 \rangle \equiv Q_1^M Q_2^N \mathbb{C}_{MN} = -\langle Q_2, Q_1 \rangle; \quad (2.2)$$

2.  $\mathbf{R}$  admits a unique rank-4 completely symmetric primitive  $G$ -invariant structure, usually named  $K$ -tensor

$$\exists! \mathbb{K}_{(MNPQ)} \equiv \mathbf{1} \in [\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}]_s; \quad (2.3)$$

thus, by contracting the  $K$ -tensor with the same charge vector  $Q$  in  $\mathbf{R}$ , one can construct a rank-4 homogeneous  $G$ -invariant polynomial, named  $\mathcal{I}_4$ :

$$\mathcal{I}_4(Q) \equiv \frac{1}{2} \mathbb{K}_{MNPQ} Q^M Q^N Q^P Q^Q, \quad (2.4)$$

which corresponds to the evaluation of the rank-4 symmetric form  $\mathbf{q}$  induced by the  $K$ -tensor on four identical modules  $\mathbf{R}$ :

$$\mathcal{I}_4(Q) = \frac{1}{2} \mathbf{q}(Q_1, Q_2, Q_3, Q_4)|_{Q_1=Q_2=Q_3=Q_4=Q} \equiv \frac{1}{2} \left[ \mathbb{K}_{MNPQ} Q_1^M Q_2^N Q_3^P Q_4^Q \right]_{Q_1=Q_2=Q_3=Q_4=Q}. \quad (2.5)$$

A famous example of *quartic* invariant in  $G = E_7$  is the *Cartan-Cremmer-Julia* invariant [20], constructed out of the fundamental irrep.  $\mathbf{R} = \mathbf{56}$ .

3. if a trilinear map  $T: \mathbf{R} \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$  is defined such that

$$\langle T(Q_1, Q_2, Q_3), Q_4 \rangle = \mathbf{q}(Q_1, Q_2, Q_3, Q_4), \quad (2.6)$$

then it holds that

$$\langle T(Q_1, Q_1, Q_2), T(Q_2, Q_2, Q_2) \rangle = \langle Q_1, Q_2 \rangle \mathbf{q}(Q_1, Q_2, Q_2, Q_2). \quad (2.7)$$

This last property makes the group of type  $E_7$  amenable to a treatment in terms of (rank-3) Jordan algebras and related Freudenthal triple systems.

Remarkably, groups of type  $E_7$ , appearing in  $D = 4$  supergravity as  $U$ -duality groups, admit a  $D = 5$  uplift to groups of type  $E_6$ , as well as a  $D = 3$  downlift to groups of type  $E_8$ ; see [21]. It should also be recalled that split form of exceptional Lie groups appear in the exceptional Cremmer-Julia [22] sequence  $E_{D(D)}$  of  $U$ -duality groups of  $M$ -theory compactified on a  $D$ -dimensional torus, in  $D = 3, 4, 5$ .

It is intriguing to notice that the first paper on groups of type  $E_7$  was written about a decade before the discovery of extended ( $N = 2$ ) supergravity [23], in which electromagnetic duality symmetry was observed [24]. The connection of groups of type  $E_7$  to supergravity can be summarized by stating that all  $2 \leq N \leq 8$ -extended supergravities in  $D = 4$  with symmetric scalar manifolds  $\frac{G}{H}$  have  $G$  of type  $E_7$  [25, 26], with the exception of  $N = 2$  group  $G = U(1, n)$  and  $N = 3$  group  $G = U(3, n)$ . These latter in fact have a quadratic invariant Hermitian form  $(Q_1, \overline{Q}_2)$ , whose imaginary part is the symplectic (skew-symmetric) product and whose real part is the symmetric quadratic invariant  $\mathcal{I}_2(Q)$  defined as follows

$$\mathcal{I}_2(Q) \equiv [\text{Re}(Q_1, \overline{Q}_2)]_{Q_1=Q_2}; \quad (2.8)$$

$$\langle Q_1, \overline{Q}_2 \rangle = -\text{Im}(Q_1, \overline{Q}_2). \quad (2.9)$$

Thus, the fundamental representations of pseudo-unitary groups  $U(p, n)$ , which have a Hermitian quadratic invariant form, do not strictly qualify for groups of type  $E_7$ .

In theories with groups of type  $E_7$ , the Bekenstein-Hawking black hole entropy is given by

$$S = \pi \sqrt{|\mathcal{I}_4(Q)|}, \quad (2.10)$$

as it was proved for the case of  $G = E_{7(7)}$  (corresponding to  $N = 8$  supergravity) in [27]. For  $N = 2$  group  $G = U(1, n)$  and  $N = 3$  group  $G = U(3, n)$  the analogue of (2.10) reads

$$S = \pi |\mathcal{I}_2(Q)|. \quad (2.11)$$

For  $3 < N \leq 8$  the following groups of type  $E_7$  are relevant:  $E_{7(7)}$ ,  $SO^*(12)$ ,  $SU(1, 5)$ ,  $SL(2, \mathbb{R}) \times SO(6, n)$ ; see Table 1. In  $N = 2$  cases of *symmetric* vector multiplets' scalar manifolds, there are 6 groups of type  $E_7$  [28]:  $E_{7(-25)}$ ,  $SO^*(12)$ ,  $SU(3, 3)$ ,  $Sp(6, \mathbb{R})$ ,  $SL(2, \mathbb{R})$ , and  $SL(2, \mathbb{R}) \times SO(2, n)$ ; see Table 2. Here  $n$  is the integer describing the number of matter (vector) multiplets for  $N = 4, 3, 2$ .

### 3 Duality Orbits

We here report some results on the stratification of the  $\mathbf{R}$  irrep. space of simple groups  $G$   $E_7$ . For a recent account, with a detailed list of Refs., see *e.g.* [29].

In supergravity, this corresponds to  $U$ -duality invariant constraints defining the charge orbits of a single-centered extremal black hole, namely of the  $G$ -invariant conditions defining the *rank* of the dyonic charge vector  $Q$  (1.8) in  $\mathbf{R}$  as an element of the corresponding Freudenthal triple system (FTS) (see [30, 31], and Refs. therein). The symplectic indices  $M = 1, \dots, \mathbf{f}$  ( $\mathbf{f} \equiv \dim_{\mathbb{R}} \mathbf{R}(G)$ ) are raised and lowered with the symplectic metric  $\mathbb{C}_{MN}$  defined by (2.1). By recalling the definition (2.4) of the unique primitive rank-4  $G$ -invariant polynomial constructed with  $Q$  in  $\mathbf{R}$ , the *rank* of a non-null  $Q$

$N$	$G$	$\mathbf{R}$
3	$U(3, n)$	$(\mathbf{3} + \mathbf{n})$
4	$SL(2, \mathbb{R}) \times SO(6, n)$	$(\mathbf{2}, \mathbf{6} + \mathbf{n})$
5	$SU(1, 5)$	$\mathbf{20}$
6	$SO^*(12)$	$\mathbf{32}$
8	$E_{7(7)}$	$\mathbf{56}$

Table 1:  $N \geq 3$  supergravity sequence of groups  $G$  of the corresponding  $\frac{G}{H}$  symmetric spaces, and their symplectic representations  $\mathbf{R}$

as an element of  $\text{FTS}(G)$  ranges from four to one, and it is manifestly  $G$ -invariantly characterized as follows:

1.  $\text{rank}(Q) = 4$ . This corresponds to “large” extremal black holes, with non-vanishing area of the event horizon (exhibiting *Attractor Mechanism* [1]):

$$\mathcal{I}_4(Q) < 0, \quad \text{or} \quad \mathcal{I}_4(Q) > 0 \quad (3.1)$$

2.  $\text{rank}(Q) = 3$ . This corresponds to “small” *lightlike* extremal black holes, with vanishing area of the event horizon:

$$\mathcal{I}_4(Q) = 0; \quad (3.2)$$

$$T(Q, Q, Q) \neq 0.$$

3.  $\text{rank}(Q) = 2$ . This corresponds to “small” *critical* extremal black holes:

$$T(Q, Q, Q) = 0; \quad (3.3)$$

$$3T(Q, Q, P) + \langle Q, P \rangle Q \neq 0.$$

4.  $\text{rank}(Q) = 1$ . This corresponds to “small” *doubly-critical* extremal BHs [14, 16]:

$$3T(Q, Q, P) + \langle Q, P \rangle Q = 0, \quad \forall P \in \mathbf{R}. \quad (3.4)$$

<b>G</b>	<b>R</b>
$U(1, n)$	<b>(1 + n)</b>
$SL(2, \mathbb{R}) \times SO(2, n)$	<b>(2, 2 + n)</b>
$SL(2, \mathbb{R})$	<b>4</b>
$Sp(6, \mathbb{R})$	<b>14'</b>
$SU(3, 3)$	<b>20</b>
$SO^*(12)$	<b>32</b>
$E_{7(-25)}$	<b>56</b>

Table 2:  $N = 2$  choices of groups  $G$  of the  $\frac{G}{H}$  symmetric spaces and their symplectic representations **R**. The last four lines refer to “magic”  $N = 2$  supergravities.

Let us consider the doubly-criticality condition (3.4) more in detail. *At least* for *simple* groups of type  $E_7$ , the following holds:

$$\mathbf{R} \times_s \mathbf{R} = \mathbf{Adj} + \mathbf{S}; \quad (3.5)$$

$$\mathbf{R} \times_a \mathbf{R} = \mathbf{1} + \mathbf{A}, \quad (3.6)$$

where **S** and **A** are suitable irreps.. For example, for  $G = E_7$  (**R** = **56**, **Adj** = **133**) one gets

$$(\mathbf{56} \times \mathbf{56})_s = \mathbf{133} + \mathbf{1463}; \quad (3.7)$$

$$(\mathbf{56} \times \mathbf{56})_a = \mathbf{1} + \mathbf{1539}. \quad (3.8)$$

For such groups, one can construct the projection operator on **Adj**( $G$ ):

$$\mathcal{P}_{AB}^{CD} = \mathcal{P}_{(AB)}^{(CD)}; \quad (3.9)$$

$$\mathcal{P}_{AB}^{CD} \frac{\partial^2 \mathcal{I}_4}{\partial Q^C \partial Q^D} = \left. \frac{\partial^2 \mathcal{I}_4}{\partial Q^A \partial Q^B} \right|_{\mathbf{Adj}(G)}; \quad (3.10)$$

$$\mathcal{P}_{AB}^{CD} \mathcal{P}_{CD}^{EF} \frac{\partial^2 \mathcal{I}_4}{\partial Q^E \partial Q^F} = \mathcal{P}_{AB}^{EF} \frac{\partial^2 \mathcal{I}_4}{\partial Q^E \partial Q^F}, \quad (3.11)$$

where (recall (3.5))

$$\frac{\partial^2 \mathcal{I}_4}{\partial Q^A \partial Q^B} = \left. \frac{\partial^2 \mathcal{I}_4}{\partial Q^A \partial Q^B} \right|_{\mathbf{Adj}(G)} + \left. \frac{\partial^2 \mathcal{I}_4}{\partial Q^A \partial Q^B} \right|_{\mathbf{S}(G)}; \quad (3.12)$$

$$\left. \frac{\partial^2 \mathcal{I}_4}{\partial Q^A \partial Q^B} \right|_{\mathbf{Adj}(G)} = 2(1 - \tau) (3\mathbb{K}_{ABCD} + \mathbb{C}_{AC}\mathbb{C}_{BD}) Q^C Q^D; \quad (3.13)$$

$$\left. \frac{\partial^2 \mathcal{I}_4}{\partial Q^A \partial Q^B} \right|_{\mathbf{S}(G)} = 2[3\tau\mathbb{K}_{ABCD} + (\tau - 1)\mathbb{C}_{AC}\mathbb{C}_{BD}] Q^C Q^D, \quad (3.14)$$



where  $\tau \equiv 2\mathbf{d}/[\mathbf{f}(\mathbf{f}+1)]$ ,  $\mathbf{d} \equiv \dim_{\mathbb{R}}(\mathbf{Adj}(G))$ . The explicit expression of  $\mathcal{P}_{AB}^{CD}$  reads<sup>3</sup> ( $\alpha = 1, \dots, \mathbf{d}$ ):

$$\mathcal{P}_{AB}^{CD} = \tau \left( 3\mathbb{C}^{CE}\mathbb{C}^{DF}\mathbb{K}_{EFAB} + \delta_{(A}^C\delta_{B)}^D \right) = -t^{\alpha|CD}t_{\alpha|AB}, \quad (3.15)$$

where the relation [34] (see also [35])

$$\mathbb{K}_{MNPQ} = -\frac{1}{3\tau}t_{(MN}^{\alpha}t_{\alpha|PQ)} = -\frac{1}{3\tau} \left[ t_{MN}^{\alpha}t_{\alpha|PQ} - \tau\mathbb{C}_{M(P}\mathbb{C}_{Q)N} \right], \quad (3.16)$$

where

$$t_{MN}^{\alpha} = t_{(MN)}^{\alpha}; \quad t_{MN}^{\alpha}\mathbb{C}^{MN} = 0 \quad (3.17)$$

is the symplectic representation of the generators of the Lie algebra  $\mathfrak{g}$  of  $G$ . Notice that  $\tau < 1$  is nothing but the ratio of the dimensions of the adjoint  $\mathbf{Adj}$  and rank-2 symmetric  $\mathbf{R} \times_s \mathbf{R}$  (3.5) reps. of  $G$ , or equivalently the ratio of upper and lower indices of  $t_{MN}^{\alpha}$ 's themselves.

## 4 From One to Two Centers

In multi-centered black hole solutions [36], a charge vector  $Q_a$  can be associated to each center, with the index  $a = 1, \dots, p$ , with  $p$  denoting the number of centers. This index transforms in the fundamental representation  $\mathbf{p}$  of the so-called “horizontal” symmetry  $SL_h(p, \mathbb{R})$  introduced in [37] (see also [38]).

We will here focus on the simplest case  $p = 2$ , presenting a number of fundamental relations defining the structure of electric-magnetic fluxes of two-centered black hole solutions [39].

From [37, 40], we define the symmetric  $\mathbf{I}_{abcd}$  tensor, sitting in the spin  $s = 2$  irrep.  $\mathbf{5}$  of  $SL_h(2, \mathbb{R})$ , as

$$\mathbf{I}_{abcd} \equiv \frac{1}{2}\mathbb{K}_{MNPQ}Q_a^MQ_b^NQ_c^PQ_d^Q. \quad (4.1)$$

Thus, its first derivative reads

$$\tilde{Q}_{M|abc} \equiv \frac{1}{4}\frac{\partial \mathbf{I}_{abcd}}{\partial Q_d^M} = \frac{1}{2}\mathbb{K}_{MNPQ}Q_a^NQ_b^PQ_c^Q = \tilde{Q}_{M|(abc)}, \quad (4.2)$$

sitting in the spin  $s = 3/2$  irrep.  $\mathbf{4}$  of  $SL_h(2, \mathbb{R})$  (the horizontal indices  $a = 1, 2$  are raised and lowered with  $\epsilon^{ab}$ , with  $\epsilon^{12} \equiv 1$ ). For clarity's sake, we report the explicit expressions of the various components of  $\mathbf{I}_{abcd}$  (4.1), as well as their relations with the components of  $\tilde{Q}_{abc}$  (4.2) (the subscripts “+2, +1, 0, -1, -2” denote the horizontal helicity of the various components [37, 40]):

$$\mathbf{I}_{+2} \equiv \mathcal{I}_4(Q_1) \equiv \mathbf{I}_{1111} = \langle \tilde{Q}_{111}, Q_1 \rangle; \quad (4.3)$$

$$\mathbf{I}_{+1} \equiv \mathbf{I}_{1112} = \langle \tilde{Q}_{111}, Q_2 \rangle = \langle \tilde{Q}_{112}, Q_1 \rangle; \quad (4.4)$$

$$\mathbf{I}_0 \equiv \mathbf{I}_{1122} = \langle \tilde{Q}_{112}, Q_2 \rangle = \langle \tilde{Q}_{122}, Q_1 \rangle; \quad (4.5)$$

$$\mathbf{I}_{-1} \equiv \mathbf{I}_{1222} = \langle \tilde{Q}_{122}, Q_2 \rangle = \langle \tilde{Q}_{222}, Q_1 \rangle; \quad (4.6)$$

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<sup>3</sup>For related results in terms of a map formulated in the “4D/5D special coordinates” symplectic frame (and thus manifestly covariant under the  $d = 5$   $U$ -duality group  $G_5$ ), see *e.g.* [32, 33].

$$\mathbf{I}_{-2} \equiv \mathcal{I}_4(Q_2) \equiv \mathbf{I}_{2222} = \langle \tilde{Q}_{222}, Q_2 \rangle. \quad (4.7)$$

Thus, one can consider the following symplectic product of spin 3/2 horizontal charge tensors:

$$\langle \tilde{Q}_{abc}, \tilde{Q}_{def} \rangle \equiv \tilde{Q}_{M|abc} \tilde{Q}_{N|def} \mathbb{C}^{MN}. \quad (4.8)$$

*A priori*,  $\langle \tilde{Q}_{abc}, \tilde{Q}_{def} \rangle$  should project onto spin  $s = 3, 2, 1, 0$  irreps. of  $SL_h(2, \mathbb{R})$  itself; however, due to the complete symmetry of the  $K$ -tensor (and to the results of [19, 34]), the projections on spin  $s = 3$  and 1 do vanish:

$$s = 3 : \langle \tilde{Q}_{(abc}, \tilde{Q}_{def)} \rangle = 0; \quad (4.9)$$

$$s = 2 : \langle \tilde{Q}_{(ab|c}, \tilde{Q}_{d|ef)} \rangle \epsilon^{cd} = \frac{2}{3} \mathcal{W} \mathbf{I}_{abef}; \quad (4.10)$$

$$s = 1 : \langle \tilde{Q}_{(a|bc}, \tilde{Q}_{de|f)} \rangle \epsilon^{bd} \epsilon^{ce} = 0; \quad (4.11)$$

$$s = 0 : \langle \tilde{Q}_{abc}, \tilde{Q}_{def} \rangle \epsilon^{ad} \epsilon^{be} \epsilon^{cf} = 8 \mathbf{I}_6, \quad (4.12)$$

where the symplectic product  $\mathcal{W}$  and the sextic horizontal polynomial  $\mathbf{I}_6$  [40] are respectively defined as (also cfr. (2.2))

$$\mathcal{W} \equiv \langle Q_1, Q_2 \rangle = \frac{1}{2} \mathbb{C}_{MN} \epsilon^{ab} Q_a^M Q_b^N; \quad (4.13)$$

$$\mathbf{I}_6 \equiv \frac{1}{8} \langle \tilde{Q}_{abc}, \tilde{Q}_{def} \rangle \epsilon^{ad} \epsilon^{be} \epsilon^{cf} = \frac{1}{4} \langle \tilde{Q}_{111}, \tilde{Q}_{222} \rangle + \frac{3}{4} \langle \tilde{Q}_{122}, \tilde{Q}_{112} \rangle. \quad (4.14)$$

The complementary relation to (4.14), namely  $\frac{1}{4} \langle \tilde{Q}_{111}, \tilde{Q}_{222} \rangle - \frac{3}{4} \langle \tilde{Q}_{122}, \tilde{Q}_{112} \rangle$  consistently turns out to be proportional (through  $\mathcal{W}$ ) to the zero helicity component of  $\mathbf{I}_{abcd}$ ; indeed, by setting  $(a, b, e, f) = (1, 1, 2, 2)$  in (4.10), one obtains:

$$\frac{1}{2} \mathbf{I}_0 \mathcal{W} = \frac{1}{4} \langle \tilde{Q}_{111}, \tilde{Q}_{222} \rangle - \frac{3}{4} \langle \tilde{Q}_{122}, \tilde{Q}_{112} \rangle. \quad (4.15)$$

We conclude by pointing out some consequences of the *rank* of a charge vector, say  $Q_1$ , on the set of two-centered invariant polynomials defined above [39]:

$$\text{rank}(Q_1) = 3 \Rightarrow \mathbf{I}_{+2} = 0; \quad (4.16)$$

$$\text{rank}(Q_1) = 2 \Rightarrow \tilde{Q}_{111} = 0 \Rightarrow \begin{cases} \mathbf{I}_{+2} = \mathbf{I}_{+1} = 0; \\ \mathbf{I}_6 = -\frac{1}{2} \mathbf{I}_0 \mathcal{W}; \end{cases} \quad (4.17)$$

$$\text{rank}(Q_1) = 1 \Rightarrow \begin{cases} \mathbf{I}_{+2} = \mathbf{I}_{+1} = 0; \\ \mathbf{I}_0 = -\frac{1}{6} \mathcal{W}^2; \\ \mathbf{I}_6 = -\frac{1}{2} \mathbf{I}_0 \mathcal{W} = \frac{1}{12} \mathcal{W}^3. \end{cases} \quad (4.18)$$

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